

Exact Solutions for the L^1TV Functional

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Outline

- L^2 TV Functional $\rightarrow L^1$ TV Functional
 - Properties of L^1 TV : $f = \chi_\Omega \Rightarrow u = \chi_\Sigma$ for some Σ .
 - Properties of L^1 TV : For $f = \chi_\Omega$, $\int |\nabla u| + \lambda \int |u - f| = \text{Per}(\Sigma) + \lambda |\Sigma \Delta \Omega|$
 - Properties of L^1 TV: $\Omega = B_{\frac{2}{\lambda}}$ and non-uniqueness.
 - Properties of L^1 TV: Preservation of contrast: for any C^2 set Ω there is a λ^* such that $\lambda \geq \lambda^*$ implies that $\Sigma = \Omega$.
 - New Result 1: $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda}} \subset \Sigma$
 - New Results 2: $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda}-\epsilon} \subset \Sigma$
 - New Result 3: A complete characterization of Σ for convex Ω .
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Motivation

We are pursuing this research with the following objectives in mind:

- To understand precise nature of denoising/reconstruction we want to understand how methods deform/preserve simple images in great detail.
 - Allows us to compare denoising methods and understand implicit biases.
 - Moves us towards a precise and systematic approach to prior informed regularization for complex experimental images.
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ROF Model and BV Functions

The classic Rudin-Osher-Fatemi (ROF) total variation regularized functional:

$$F(u) \equiv \int |\nabla u| dx + \lambda \int |u - f|^2 dx \quad (1)$$

is characterized by *simultaneous edge recovery/preservation and noise reduction* but also *loss of contrast*.

(I) *total variation of $u = \int |\nabla u|$*

(II) $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and $\int |\nabla u| < \infty$.

$$(III) \int |\nabla u| = \sup \left\{ \int \nabla u \cdot \vec{g} ; |\vec{g}| < 1, \vec{g} \in C_0^1(\Omega; \mathbb{R}^n) \right\} \quad (2)$$

$$= \sup \left\{ \int u \operatorname{div} \vec{g} ; |\vec{g}| < 1, \vec{g} \in C_0^1(\Omega; \mathbb{R}^n) \right\} \quad (3)$$

for $u \in W^{1,1}$.

ROF Model and BV Functions: cont.

- (IV) For $u \in L^1(\Omega)$, we use the last equation to *define* $\int |\nabla u|$
- (V) The theory for $BV(\Omega)$ is extensive and quite beautiful [see for example Giusti's "Minimal Surfaces and Functions of Bounded Variation" and Evan's and Gariepy's "Measure Theory and Fine Properties of Functions"]

Other facts:

1. $TV(u)$ is lower semi-continuous in L^1
 2. Approximation, compactness, and trace results are similar to Sobolev spaces.
 3. $Du = \nabla u$ is a vector valued Radon measure
 4. $\int |\nabla u| = \int_{u_{\min}}^{u_{\max}} \text{Per}(\{x ; u(x) > t\}) dt$ (coarea formula)
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L^1 TV Functional: $TV + L^1$ Data Fidelity

The L^1 TV functional, previously studied by [Alliney], [Nikolova] and [Chan and Esedoglu] is given by:

$$F(u) \equiv \int |\nabla u| dx + \lambda \int |u - f| dx \quad (4)$$

and is characterized by *simultaneous edge recovery/preservation and noise reduction without the loss of contrast*. But this is not all:

1. $F(u)$ is not strictly convex \Rightarrow we do not have uniqueness!
2. u is a minimizer for $f \rightarrow Cu$ is a minimizer for Cf
3. $f = \chi_{\Omega} \rightarrow u = \chi_{\Sigma}$.

We now look at the properties of L^1 TV a bit more closely.

L^1 TV: $f = \chi_{\Omega} \Rightarrow u = \chi_{\Sigma}$ for some Σ

Result: [Chan and Esedoglu] If u is any minimizer of $F_{\lambda}(u)$ then for almost all $\mu \in [0, 1]$,

$$\chi_{\{x:u>\mu\}}$$

is also a minimizer of $F_{\lambda}(u)$.

One Dimensional Example: λ determines which interval of Ω appears in Σ .

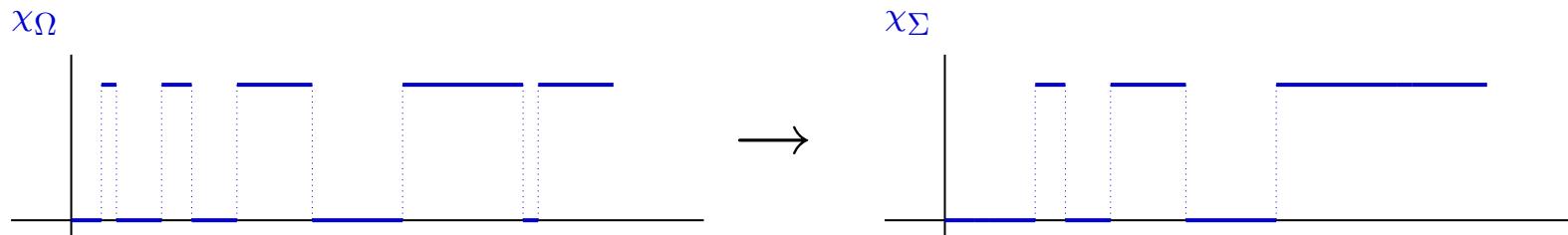
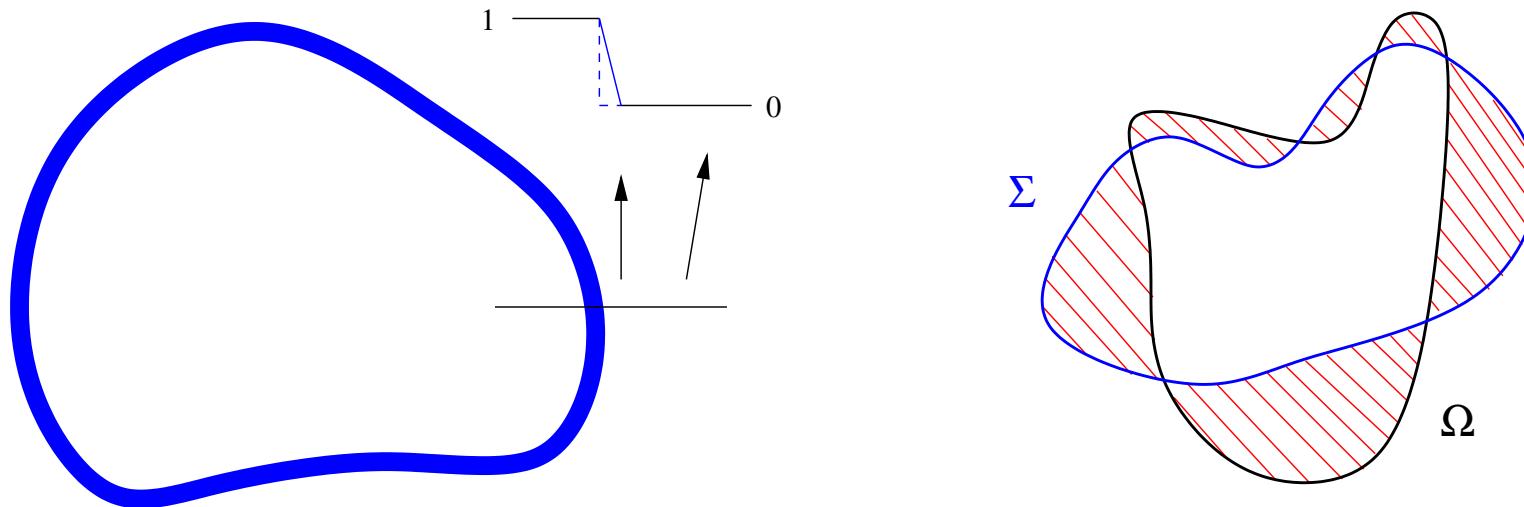


Figure 1: Small segments disappear: λ determines “small”

Segment preserved if $\{\text{Per}(I) = 2 < \lambda L_I \Leftrightarrow L_I > \frac{2}{\lambda}\}$.

$$L^1 \text{ TV: } f = \chi_{\Omega} \rightarrow F(u) = F(\Sigma) = \text{Per}(\Sigma) + \lambda |\Sigma \Delta \Omega|$$

For characteristic functions $u = \chi_{\Sigma}$ (binary images) $\int |\nabla u|$ is exactly the perimeter and $\lambda \int |u - f| = \lambda \int |\chi_{\Sigma} - \chi_{\Omega}| = \lambda \int |\Sigma \Delta \Omega|$.



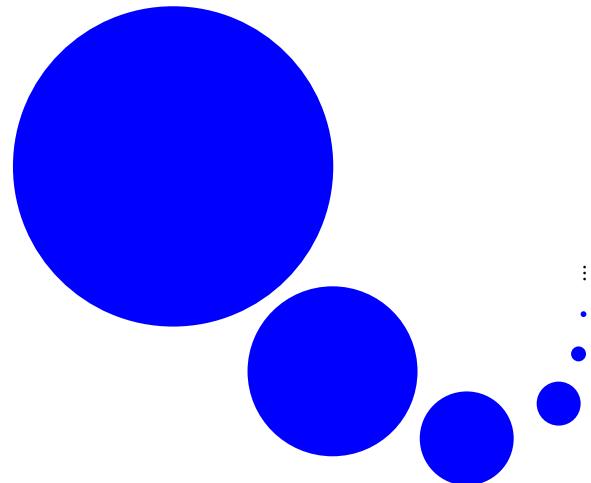
$$\int |\nabla \chi_{\Sigma}| = \text{Per}(\Sigma)$$

$$\lambda \int |u - f| = \lambda \int |\chi_{\Sigma} - \chi_{\Omega}| = \lambda \int |\Sigma \Delta \Omega|$$

L^1 TV: $\Omega = B_{\frac{2}{\lambda}}$ and non-uniqueness.

Result: [Chan and Esedoglu] If $\Omega = B_{\frac{2}{\lambda}}$ then $u = \alpha \chi_{B_{\frac{2}{\lambda}}}$ is a minimizer for any $\alpha \in [0, 1]$.

One can therefore concoct Ω 's whose solutions $\Sigma(\lambda)$ have, as $\lambda \rightarrow \infty$, an infinite number of non-uniqueness points ...



(The points in λ at which this non-uniqueness is easy to concoct are the points of discontinuity of the monotonically increasing “function” $\|u(\lambda) - f\|_1$ and there are at most a countable number of such points.)

L^1 TV: smooth Ω + big λ imply $\Sigma = \Omega$.

The previous example demonstrated an Ω that is never reproduced by Σ as $\lambda \rightarrow \infty$. When Ω is a bounded, C^2 set this can't happen:

Result:[Chan and Esedoglu] For Ω bounded and C^2 there is a $\lambda < \infty$ such that for all $\lambda^* \geq \lambda$, $\Sigma = \Omega$.

Choose \vec{g} such that $\int_{\Omega} 1 \operatorname{div} \vec{g} = \operatorname{Per}(\Omega)$:

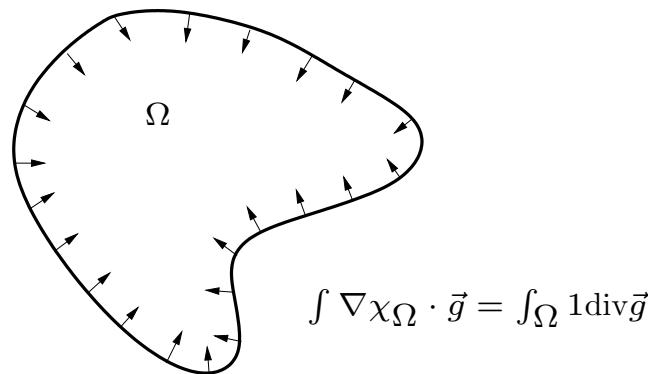


Figure 2: choosing a vector field

L^1 TV: smooth Ω + big λ imply $\Sigma = \Omega$.

$$\text{Per}(\Sigma) + \lambda |\Sigma \Delta \Omega| \geq \int_{\Sigma} 1 \operatorname{div} \vec{g} + \lambda \int_{\Omega \cap \Sigma^c} 1 + \lambda \int_{\Omega^c \cap \Sigma} 1 \quad (5)$$

$$= \int_{\Omega \cap \Sigma} 1 \operatorname{div} \vec{g} + \int_{\Omega^c \cap \Sigma} 1 \operatorname{div} \vec{g} + \lambda \int_{\Omega \cap \Sigma^c} 1 + \lambda \int_{\Omega^c \cap \Sigma} 1 \quad (6)$$

$$\geq \int_{\Omega \cap \Sigma} 1 \operatorname{div} \vec{g} + \int_{\Omega^c \cap \Sigma} 1 \operatorname{div} \vec{g} + \int_{\Omega \cap \Sigma^c} 1 \operatorname{div} \vec{g} + \lambda \int_{\Omega^c \cap \Sigma} 1 \quad (7)$$

$$= \int_{\Omega} 1 \operatorname{div} \vec{g} + \int_{\Omega^c \cap \Sigma} 1 (\lambda + \operatorname{div} \vec{g}) \quad (8)$$

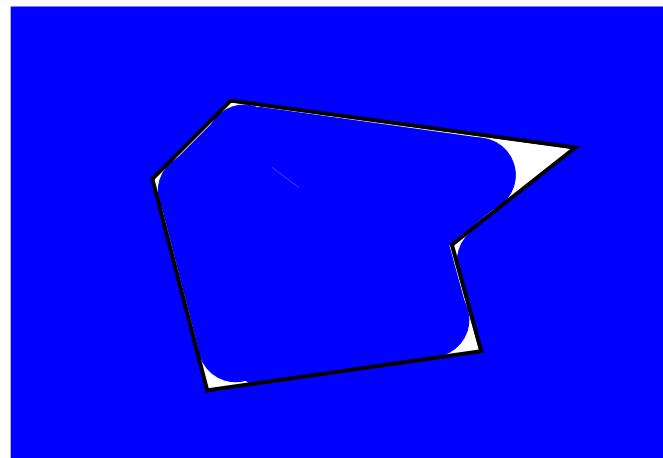
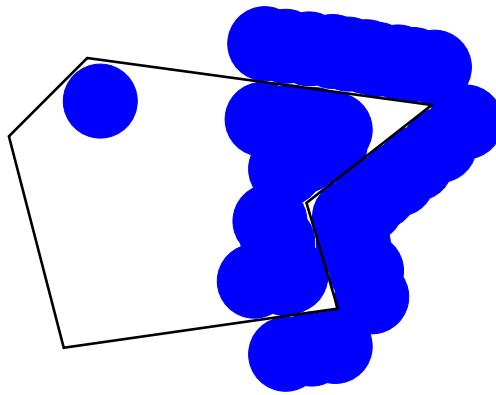
$$\geq \text{Per}(\Omega) \quad (9)$$

as long as $\lambda > \|\operatorname{div} \vec{g}\|_{\infty}$.

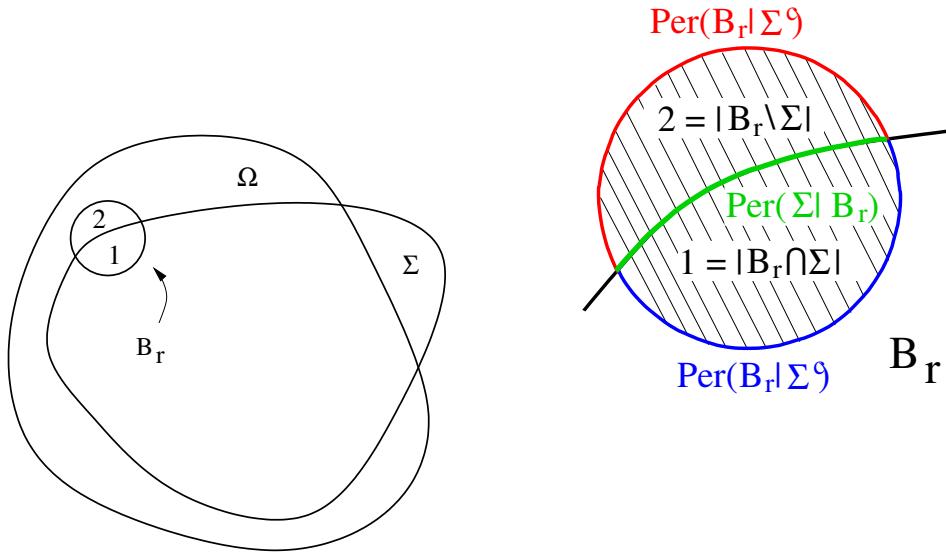
New Result 1: $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda}} \subset \Sigma$

Theorem 1. *If $B_r \subset \Omega$ where $r \geq \frac{2}{\lambda}$, then $B_r \subset \Sigma$.*

In particular, we can conclude that the boundary of Σ is in the envelope of inside and outside $\frac{2}{\lambda}$ balls.



New Result 1: $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda}} \subset \Sigma$

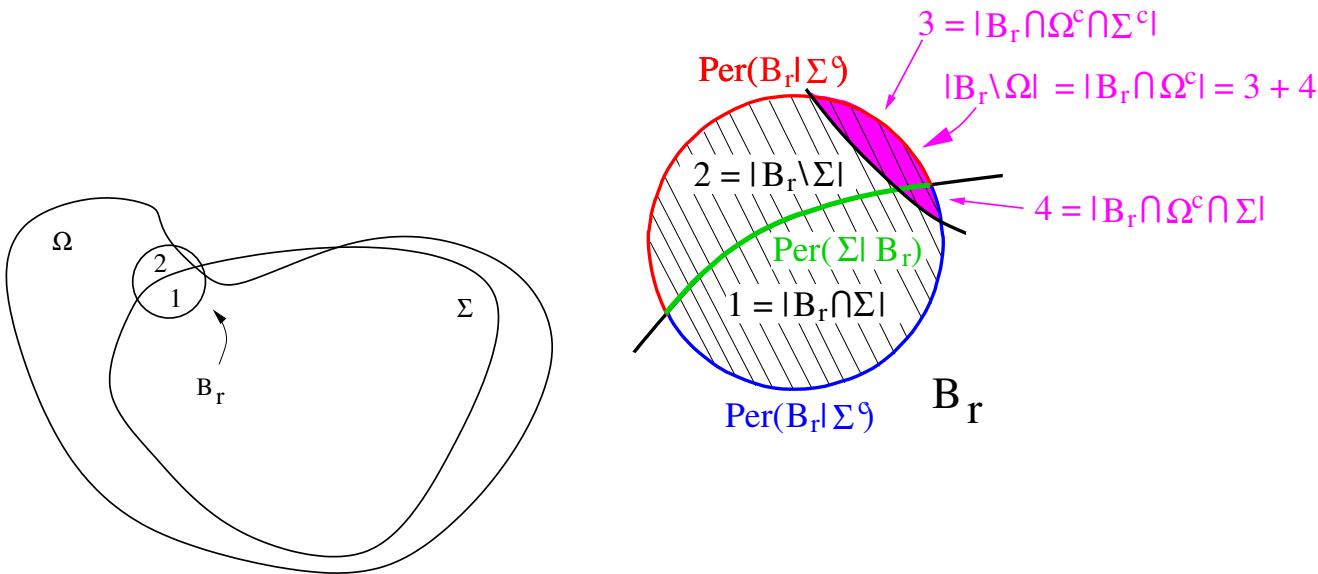


$$E(\Sigma \cup B_r) - E(\Sigma) = (\text{Per}(B_r) - \lambda|B_r|) + (\lambda|B_r \cap \Sigma| - \text{Per}(B_r \cap \Sigma)) \quad (10)$$

$$= \left(2\pi r - \frac{2}{R}\pi r^2 \right) + \left(\frac{2}{R}\pi\rho^2 - 2\pi\rho^* \right) \quad (11)$$

$$= 2\pi r \left(1 - \frac{r}{R} \right) + 2\pi\rho \left(\frac{\rho}{R} - \frac{\rho^*}{\rho} \right) \quad (12)$$

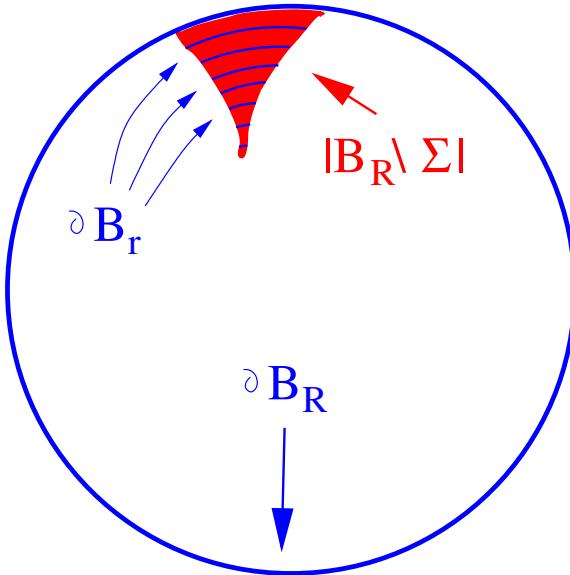
New Result 2: $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda} - \epsilon} \subset \Sigma$



$$E(\Sigma \cup B_r) - E(\Sigma) \leq 2\pi r \left(1 - \frac{r}{R}\right) + 2\pi \rho \left(\frac{\rho}{R} - \frac{\rho^*}{\rho}\right) + 2\lambda |B_r \setminus \Omega| \quad (13)$$

Theorem 2. Given a ball $B_{\hat{r}}$ with $\frac{2}{\lambda} < \hat{r} < \frac{5}{\lambda}$ and an ϵ such that $(1 - \frac{1}{\sqrt{2}}) > \epsilon > 0$, we can choose a $\theta > 0$ such that if $|B_r \setminus \Omega| < \theta |B_{\hat{r}}|$, then $B_{r^*} \subset \Sigma$ for $r^* = (1 - \epsilon) \frac{2}{\lambda}$.

Idea of proof: A Gronwall inequality and Comparisons



$$E(\Sigma \cup B_r) - E(\Sigma) \leq -\text{Per}(\Sigma; B_r(x_0)) + \text{Per}(B_r; \Sigma^c) + \lambda |B_r \setminus \Sigma| \quad (14)$$

$$\leq -C\sqrt{v(r)} + \dot{v}(r) + \lambda v(r) \quad (v(r) \equiv |B_r \setminus \Sigma|) \quad (15)$$

$$0 \leq -C\sqrt{v(r)} + \dot{v}(r) + \lambda v(r) \Rightarrow \sqrt{v(r)} \leq \frac{CR}{2} \left(e^{-\frac{R-r}{R}} - 1 \right) + \sqrt{|B_R \setminus \Sigma|}$$

⇒ small enough $|B_R \setminus \Sigma| \Rightarrow v(R - \epsilon) = 0 \Rightarrow B_{R-\epsilon} \subset \Sigma$.

$$|B_R \setminus \Omega| \text{ small} \stackrel{?}{\Rightarrow} |B_R \setminus \Sigma| \text{ small}$$

Idea of proof:

The rest of the proof is a fairly intricate argument showing that when:

$$|B_R \setminus \Omega| < \delta$$

then of the three cases:

$$1 \ |B_R \setminus \Sigma| \leq N\delta$$

$$2 \ N\delta \leq |B_R \setminus \Sigma| < \frac{1}{4}\pi R^2$$

$$3 \ \frac{1}{4}\pi R^2 \leq |B_R \setminus \Sigma|$$

only case 1 occurs. This is obtained by making use of:

$$E(\Sigma \cup B_r) - E(\Sigma) \leq 2\pi r(1 - \frac{r}{R}) + 2\pi\rho(\frac{\rho}{R} - \frac{\rho^*}{\rho}) + 2\lambda|B_r \setminus \Omega| \quad (16)$$

New Result 3: Exact Σ for any convex Ω

Theorem 3. *Using a result of Allard's, we can conclude that for convex Ω , $\Sigma =$ the union of all $\frac{1}{\lambda}$ balls which are contained in Ω PROVIDED there is at least one $\frac{2}{\lambda}$ ball contained in Ω .*

Outline of Proof:

If Ω is convex and Σ (which must be contained in Ω) is not empty, then Σ is the union of the $\frac{1}{\lambda}$ balls in Ω . Our result says that if Ω contains a $\frac{2}{\lambda}$ ball, then it is contained in a solution Σ . Therefore, using Allard's result, Σ must equal the union of $\frac{1}{\lambda}$ balls in Ω .

Comments and Conclusions

- To Do: Establish connections to morphology – opening and closing, etc.
 - To Do: Exact solutions with noise – further results.
 - To Do: Understand the regularization and reconstruction aspects image analysis for experimental data in which physics is partly understood and partly being explored.
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